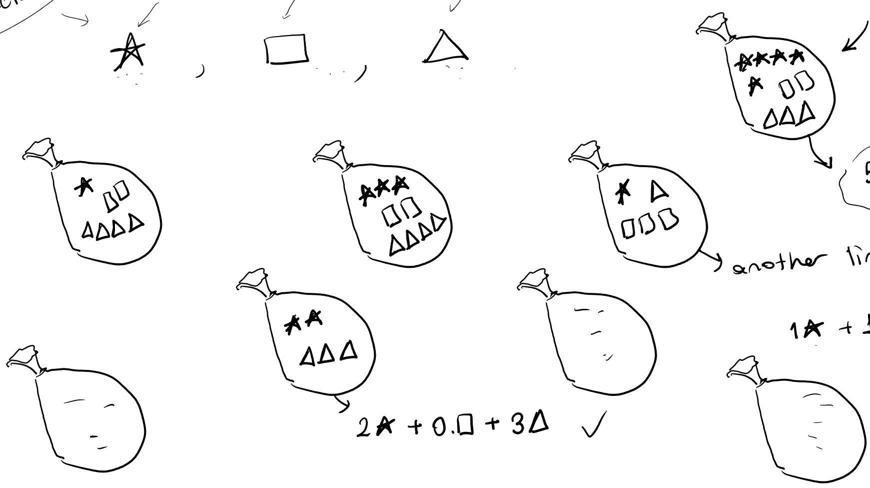
8th Week Friday

14 Nisan 2023 Cuma 10:32

Linear Combination & Span

vectors → ★ , □ , △



a linear combination of ★, □, △

5★ + 2□ + 3△

another linear combination

1★ + 1△ + 3□

2★ + 0.□ + 3△ ✓

Span : The set of all linear combinations of vectors.

$$\text{Span}(\{★, □, △\}) = \{ \underbrace{r_1 ★ + r_2 □ + r_3 △}\ :\ r_1, r_2, r_3 \in \mathbb{R} \}$$

Ex/ $\mathbb{R}^3$

$(-2,3,4)$

$v_1 = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} \in \mathbb{R}^3$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$

$$(-3).v_1 + 2.v_2 = (-3)\begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} + (2)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ -6 \end{bmatrix}$$

a linear combination of vectors v_1, v_2

a) Can $\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$ be written as a linear combination of v_1 and v_2?) = ✓

$$r_1 \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} + r_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$$

Can we find real numbers r_1, r_2 satisfying this eqn?

$$\begin{bmatrix} -2r_1 \\ 3r_1 \\ 4r_1 \end{bmatrix} + \begin{bmatrix} r_2 \\ 2r_2 \\ 3r_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} \Rightarrow \begin{aligned} -2r_1 + r_2 &= 4 \\ 3r_1 + 2r_2 &= 5 \\ 4r_1 + 3r_2 &= -6 \end{aligned}$$

$$\left[\begin{array}{cc|c} -2 & 1 & 4 \\ 3 & 2 & 5 \\ 4 & 3 & -6 \end{array}\right] \xrightarrow{-\frac{1}{2}r_1 \to r_1} \left[\begin{array}{cc|c} 1 & -1/2 & 2 \\ 3 & 2 & 5 \\ 4 & 3 & -6 \end{array}\right] \xrightarrow[-4r_1 + r_3 \to r_3]{-3r_1 + r_2 \to r_2} \left[\begin{array}{cc|c} 1 & -1/2 & 2 \\ 0 & 7/2 & -1 \\ 0 & 5 & -14 \end{array}\right]$$

$\frac{3}{2} + 2$

$$\xrightarrow{\frac{2}{7}r_2 \to r_2} \left[\begin{array}{cc|c} 1 & -1/2 & 2 \\ 0 & 1 & -2/7 \\ 0 & 5 & -14 \end{array}\right] \xrightarrow{-5r_2 + r_3 \to r_3} \left[\begin{array}{cc|c} 1 & -1/2 & 2 \\ 0 & 1 & -2/7 \\ 0 & 0 & \frac{10}{7} - 14 \end{array}\right] \to \text{No soln!}$$

$\neq 0$

$2 +$

$\frac{10}{7} - 14$

other special subspaces related to matrices

null space of $A_{m\times n}$
$N(A) \leq \mathbb{R}^n$ ✓

Row Space of $A_{m\times n} \leq \mathbb{R}^n$

$\in \mathbb{R}^n$ A $m\times n$

Column Space of $A \leq \mathbb{R}^m$

$\in \mathbb{R}^m$ $m\times n$

Row Space of $A_{m\times n}$: $R(A)$

A N(A) R(A) C(A)

$$A \longrightarrow \text{REF} \longrightarrow \text{Span}\left\{\begin{array}{l}\text{Take all the} \\ \text{not-all-zero rows} \\ \text{of REF}\end{array}\right\} = R(A)$$

Column Space of $A_{m\times n}$: $C(A)$

$A \longrightarrow$ REF $\longrightarrow$ look at column numbers of leading 1's $\longrightarrow$ $\text{Span}\left\{\begin{array}{l}\text{take the columns of } A \\ \text{in these positions}\end{array}\right\} = C(A)$

1st 5th

$$\underbrace{\begin{bmatrix} 1 & & 1 & \\ & & & \end{bmatrix}}_{\text{REF}(A)}$$

Ex/

(d) $A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{bmatrix}$

$N(A) = ?$ → $Ax = 0$ $R(A) = ?$ $C(A) = ?$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 2 & 2 & -3 & 1 & 0 \\ -1 & -1 & 0 & -5 & 0 \end{array}\right] \xrightarrow[r_1 + r_3 \to r_3]{-2r_1 + r_2 \to r_2} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & -1 & -3 & 0 \\ 0 & 0 & -1 & -3 & 0 \end{array}\right]$$

$-1r_2 \to r_2$ ~

$$\xrightarrow{r_2 + r_3 \to r_3} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \underbrace{\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{REF}(A)} \end{array}$$

$x_2 = r \in \mathbb{R}$
$x_4 = s \in \mathbb{R}$
$x_3 = -3s$
$x_1 = -r - 5s$

$x_1 + r + 3s + 2s = 0$

$N(A) = \{(-r-5s, r, -3s, s) : r, s \in \mathbb{R}\}$

r, s, t ..

$$\begin{bmatrix} -r-5s \\ r \\ -3s \\ s \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

? ?

$$N(A) = \text{Span}\left(\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix}\right\}\right)$$

$R(A) = ?$

$R(A) = \text{Span}\left(\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}\right) \leq \mathbb{R}^4$

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{REF(A)} \Rightarrow R(A) = \text{Span}\left(\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}}_{v_2} \right\}\right) \leq \mathbb{R}^4$$

$$= \{ r v_1 + s v_2 : r, s \in \mathbb{R} \}$$

$C(A) = ?$

$$\begin{array}{cccc} 1^{st} & & 3^{rd} & \\ \downarrow & & \downarrow & \\ \end{array}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{REF(A)}$$

A

$$A = \begin{array}{cccc} v_1 & & v_2 & \\ \downarrow & & \downarrow & \\ \end{array}$$

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{bmatrix}$$

$$C(A) = \text{Span}\left(\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} \right\}\right) \leq \mathbb{R}^3$$

$C(A) = \text{Span}(\{v_1, v_2\})$

$C(A) = \{ r v_1 + s v_2 : r, s \in \mathbb{R} \}$

! Span is not uniquely determined.